Moments conservation in adaptive Vlasov solver

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Abstract

We previously developed an adaptive semi-Lagrangian solver using a multiresolution analysis based on interpolets which are a kind of interpolating wavelets introduced by Deslauriers and Dubuc. This paper introduces a new multiresolution approximation for this solver which allows to conserve moments up to any order in the thresholding step by using the lifting method introduced by Sweldens.

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The model we consider throughout this paper is the nonrelativistic Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0, \tag{1}$$

where the self electric field $\mathbf{E}$ is computed from Poisson's equations. The magnetic field is external and considered to be known.

The numerical solution of the Vlasov equation is usually performed by particle-in-cell methods (PIC) which are known to suffer from numerical noise. To remedy this problem and obtain a more accurate description of the distribution function, methods discretizing the Vlasov equation on a mesh of phase space have been proposed [1,4,5]. In order to avoid the high numerical cost of such methods using a uniform and fixed mesh, we develop adaptive methods.

Our adaptive method is overlaid on a classical semi-Lagrangian method. Adaptivity, that is refinement or derefinement of the mesh, is based on a multiresolution analysis (section 1, see [3] for more details). In the present work, we focus on moment conservation for the Vlasov equation during the thresholding step. This ensures in particular that the total number of particles is conserved. We explain (section 2) how to use the lifting procedure introduced by W. Sweldens [6] in order to ensure this conservation of moments. Relevant numerical results are presented in section 3.

1 Multiresolution analysis

The semi-Lagrangian method consists in computing point values of the distribution function on a grid of phase-space. It consist of two steps, an advection step needed to determine the origin of the characteristics ending at the grid points and an interpolation step which is used to compute the value of the distribution function at these points. The multiresolution approximation will be used to minimize the number of interpolation points for a given approximation error.

First, we define an infinite sequence of nested grids $(G_j)_{j\in\mathbb{Z}}$, where j is called the level of the grid, the grid points being located at $x_k^j = k2^{-j}$. In order to define an adaptive grid, we want to compare a function f defined by its values $(c_k^{j+1} = f(x_k^{j+1}))_{k\in\mathbb{Z}}$ on a finer grid G_{j+1} to its restriction $(c_k^j = f(x_k^j) = f(x_{2k}^{j+1}) = c_{2k}^{j+1})_{k\in\mathbb{Z}}$ on a coarser grid G_j. For this purpose, we need a prediction operator to define an approximation of $f(x_{2k+1}^{j+1}) = c_{2k+1}^{j+1}$ from the (c_k^j). Using an odd degree Lagrange interpolation polynomial P_{2N-1}, this approximation is given by $P_{2N-1}(x_{2k+1}^{j+1})$ and the approximation errors, called details in wavelet terminology, are given by

$$d_k^j = c_{2k+1}^{j+1} - P_{2N-1}(x_{2k+1}^{j+1}) = c_{2k+1}^{j+1} - \sum_{n=1-N}^{N} a_n c_{2k+2,}^{j+1}. \tag{2}$$

This can be formulated in the framework of biorthogonal wavelets introduces by Cohen, Daubechies and Feauveau [2]. The above decomposition of f can also be expressed using basis functions φ, named scaling function, and ψ, named wavelet, such that $\psi(x) = \phi(2x-1)$:

$$f(x) = \sum_k c_k^{j+1}\varphi_k^{j+1}(x) = \sum_k c_k^j \varphi_k^j(x) + \sum_k d_k^j \psi_k^j(x), \tag{3}$$

where $\varphi_k^j(x) = \varphi(2^j x - k)$ and $\psi_k^j(x) = \psi(2^j x - k)$. For interpolating polynomials P_{2N-1}, the values a_n defined in (2) are $a_0 = a_1 = \frac{1}{2}$ and else $a_n = 0$ for degree 1, and $a_{-1} = a_2 = -\frac{1}{16}$, $a_0 = a_1 = \frac{9}{16}$ and else $a_n = 0$ for degree 3. The scaling function φ and wavelet ψ for the corresponding values of N are displayed in Figures 1 and 2.

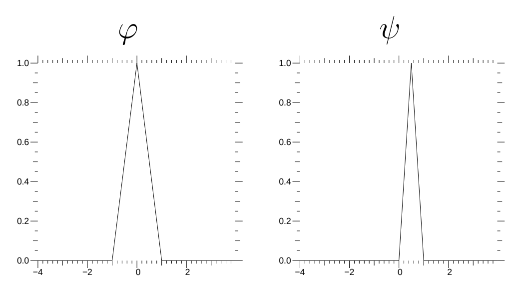

Fig. 1. Scaling functions and wavelets for linear interpolation polynomials ($N = 1$).

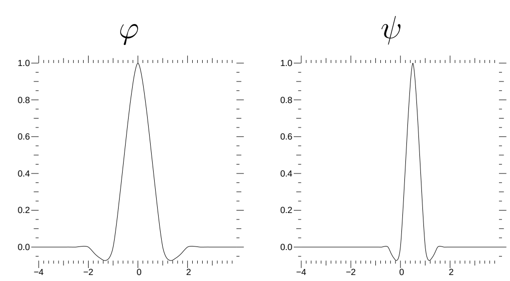

Fig. 2. Scaling functions and wavelets for cubic interpolation polynomials ($N = 2$).

We now consider the thresholding. From (2), we note that the improvement of approximation is locally important around grid point $2^{-j}(2k+1)$ when the detail d_k^j

is large, and conversely small when the detail is small. Hence the representation of a function at level $j+1$ can be compressed with a controled approximation loss by setting to 0 the details with an absolute value less than some given threshold $\epsilon_j > 0$ depending on the level j.

Thus, in the decomposition formula (3), we eliminate terms $d_k^j \psi_k^j$ such that $|d_k^j| < \epsilon_j$ and we can bound the error commited because of this thresholding process as follows

$$\|e_{j+1}^s\|_{L^p} = \Big\| \sum_{k \,|\, d_k^j < \epsilon_j} d_k^j \psi_k^j \Big\|_{L^p} \leq \sum_{k \,|\, |d_k^j| < \epsilon_j} |d_k^j| \|\psi_k^j\|_{L^p} \leq \epsilon_j \sum_{k \,|\, |d_k^j| < \epsilon_j} 2^{-\frac{j}{p}} \|\psi\|_{L^p} \tag{4}$$

$$\leq \epsilon_j 2^{-\frac{j}{p}} \|\psi\|_{L^p} \, \#\Big(\{k \mid |d_k^j| < \epsilon_j\}\Big), \tag{5}$$

since, assuming that f_{j+1} has compact support, the number of removed terms is finite. In (4-5), $\|\psi\|_{L^p} = (\int_{\mathbb{R}} |\psi(x)|^p \, dx)^{1/p}$ for $p \geq 1$ is the norm in which the error is measured.

2 Conservation of moments

When numerically solving the Vlasov equation, it is often essential to conserve the density of particles. Moreover, in order to get a better accuracy for the thresholded distribution function, it can be helpful to conserve higher order moments while performing adaptivity. Since the thresholding procedure consists in removing linear combinations of the ψ_k^j, this procedure conserves moments if the corresponding moments of ψ that are $\int x^p \psi(x) \, dx$ vanish.

We saw that the wavelet associated to the interpolating scaling function is $\psi(x) = \varphi(2x-1)$. Hence, as the moments of φ do not vanish, the moments of ψ do not vanish. However, the lifting procedure introduced by Sweldens [6] can be used to define a new set of biorthogonal scaling functions and wavelets from a given one so that some desired properties are satisfied.

Consider the decomposition formula (3). Following Swelden's construction, we can modify the basis (ψ_j^k) without modifying the multiresolution approximation by taking a new wavelet $\bar{\psi}$ of the following form

$$\bar{\psi}(x) = \psi(x) - \sum_k s_k \, \varphi(x-k), \tag{6}$$

where the coefficients $(s_k)_k$ define the new wavelet.

To conserve density, one needs $\int \bar{\psi}(x) \, dx = 0$ which yields

$$0 = \int \varphi(2x-1) \, dx - \sum_k s_k \int \varphi(x-k) \, dx = \left(\frac{1}{2} - \sum_k s_k\right) \int \varphi(x) dx. \tag{7}$$

Thus we need to ensure that $\sum_k s_k = 1/2$. To keep the symmetry of $\bar{\psi}$, we also impose that $s_k = s_{1-k}$. Then the simplest choice for $(s_k)_k$ is $s_0 = s_1 = 1/4$ and $s_k = 0$ else. With this choice of s_0 and s_1, $\bar{\psi}$ is an even function so that the first order moment is also conserved.

More generally, we can compute the s_k in order to conserve the moment of any even order, always keeping the symmetry ensuring $s_k = s_{1-k}$. Then if all the moments of even order are conserved up to some even n, all the moments of any odd order are also conserved up to $n + 1$.

In figure 3 and table 1, we respectively construct the corrected wavelet in order to ensure density conservation and give the simplest choice of s_k in order to ensure the conservation of the second order moment.

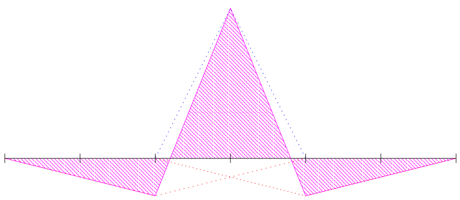

Fig. 3. Correction of linear wavelet in order to conserve density.

linear wavelets: $s_0 = s_1 = \frac{19}{64}, \quad s_{-1} = s_2 = \frac{-3}{64}$

cubic wavelets: $s_0 = s_1 = \frac{149}{512}, \quad s_{-1} = s_2 = \frac{-21}{512}$

Table 1
Simplest choice of s_k for second order moment conservation.

3 Adaptive algorithm and numerical result

The wavelet decomposition we introduced yields an adaptive algorithm consisting of the following step: starting from a compressed distribution function f^n defined on an adaptive grid, we predict the set of grid points that will contain the next adaptive grid by advecting the grid points along the characteristics. Then, we perform the semi-Lagrangian algorithm on the predicted points and finally compute the wavelet decomposition of f^{n+1} to get the new set of important details and the corresponding adaptive grid (see [3] for more details on the algorithm).

In order to assess the benefits of the adaptive solver we computed the transverse evolution of a semi-Gaussian beam in a periodic focusing field of the form $\alpha(1 + \cos 2\pi z/S)$ for a tune depression σ/σ_0 of 0.17. The initial distribution function reads

$$f(r, v) = \frac{I}{\pi a^2 \sqrt{2\pi} b} e^{-\frac{1}{2}(v^2/b^2)} \text{ if } r < a, \qquad \text{and } f(r, v) = 0 \text{ else.} \tag{8}$$

In order to show the importance of the moments conservation, we computed the evolution of a two-stream instability. The initial function is given by:

$$f_0 = \frac{1}{\sqrt{2\pi}} v^2 \exp(-v^2/2)(1 + \alpha cos(kx)), \tag{9}$$

with $\alpha = 0.05$ and $k = 0.5$. The physical mesh is $x-$periodic: $[0, 4\pi[\times [-6, 6[$.

We notice in Figure 4 that the adaptive grid follows very well the evolution of the fine structures. In Figure 5, we notice that ensuring density conservation avoided unstability problems at the expense of a slightly higher diffusion and computational cost when we use the lifting procedure.

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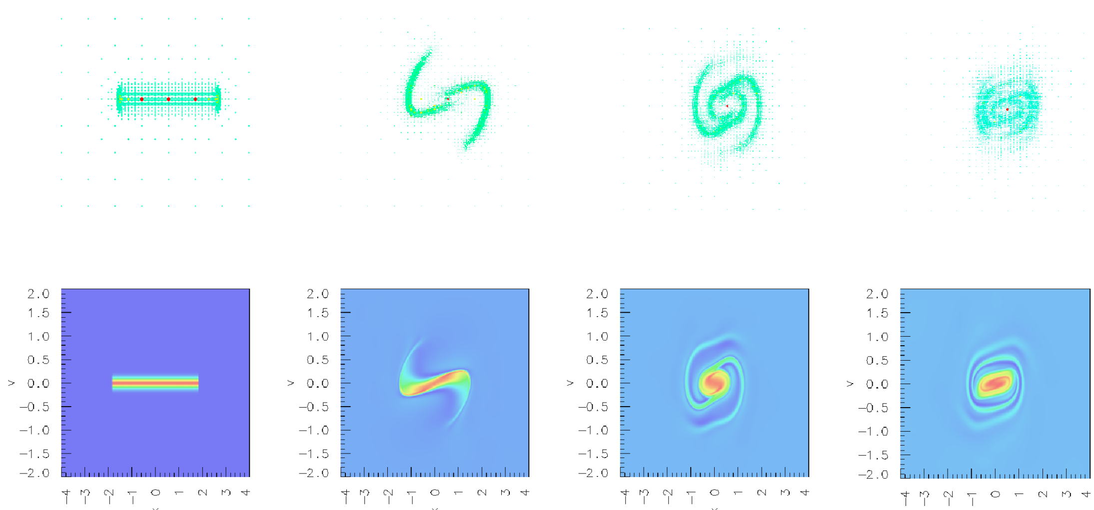

Fig. 4. Semi-Gaussian beam and the associated adaptive grid in a periodic focusing channel at 0, 15, 30 and 45 ω_p^{-1}.

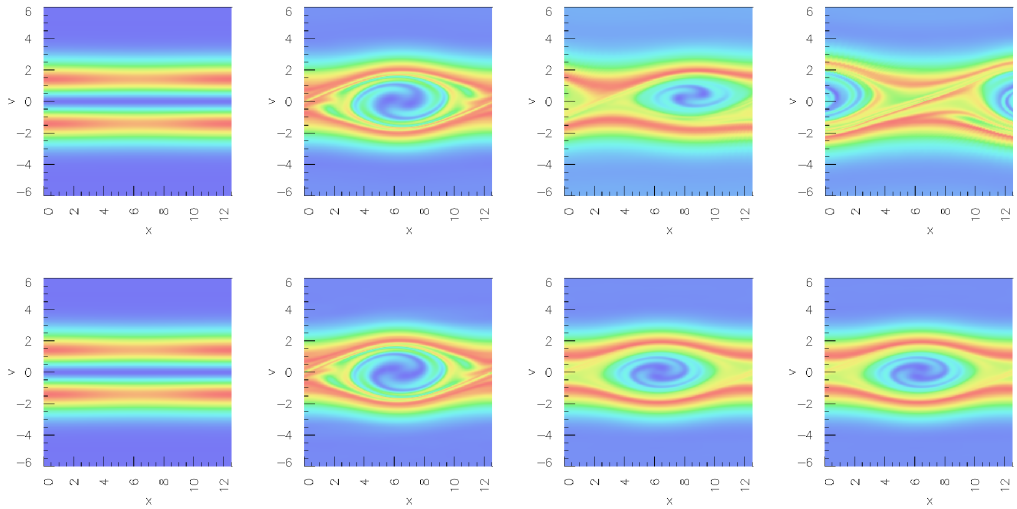

Fig. 5. Evolution of a two stream instability without (top) and with (bottom) density conservation at 0, 37.5, 75 and 112.5 ω_p^{-1}.